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Topographic waves in open domains. Part 1. Boundary conditions and frequency estimates

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The problem is considered of topographic waves propagating on depth gradients in rotating domains. A variational principle is derived for the eigenfunctions and eigenfrequencies of normal modes on a domain, and applied to subdomains of the whole domain. Considering suitable boundary conditions on the open boundary between the subdomain and the remainder of the whole domain gives upper and lower bounds and estimates for the frequencies of normal modes localized in the subdomain without the complication of solving over the whole domain. It is shown that applying a zero-mass-flux condition at the open boundary leads to a lower bound on the frequency whereas requiring a particular form of the energy flux to vanish identically at each point of the boundary provides an upper bound.

1. Introduction

Interest in topographic waves has been concerned in the main with coastal shelf waves propagating above depth profiles that are constant or vary only slowly with along-shore displacement (Huthnance 1975; Mysak 1980). Recent observations in the Lake of Lugano (Mysak *et al.* 1985) and their numerical modelling (Trösch 1984; Stocker & Hutter 1987*a*, *b*) show, however, that disturbances can be highly localized (bay modes). In both the finite-difference calculations of Trösch and the modal expansion and shooting calculations of Stocker & Hutter, much of the numerical effort is expended on evaluating the solutions in parts of the domain where the associated velocities are negligible. Computational efficiency would be increased greatly if attention were confined to a small region with an appropriate open boundary condition replacing the remainder of the domain. It is the purpose of the present paper to give a method for obtaining rigorous bounds on the frequencies of normal modes by considering different open boundary conditions.

Section 2 introduces a variational principle for the eigenfunctions and eigenfrequencies of a given domain and demonstrates the orthogonality of bay modes. Section 3 considers the principle applied to a subdomain of the original region. It is shown that lower bounds for the eigenfrequencies are obtained by applying a zero-mass-flux condition on the open boundaries. Upper bounds follow from applying the natural boundary condition for the variational principle. Both these conditions imply that there is no energy flux across the open boundary. For open boundaries coincident with a contour of f/H, the natural boundary condition reduces to requiring the tangential velocity to vanish. Section 3 includes a simple illustration of the accuracy of the bounds in the case of an axisymmetric island with a skirt. Further applications are given in Part 2 (Stocker & Johnson 1989*a*) for a semi-infinite channel with a bay zone at the closed end. Section 4 discusses implementation of the results in numerical analyses and their use in geophysical contexts.

2. A variational principle

The linearized, barotropic, non-divergent potential vorticity equation or topographic wave equation can be written

$$\nabla \cdot (H^{-1}\nabla \Psi_t + \Psi \boldsymbol{K}) = 0, \tag{2.1}$$

where Ψ is the depth-integrated volume flux stream function, H is the local depth, f is the (variable) Coriolis parameter, $\boldsymbol{K} = \hat{z} \wedge \nabla(f/H)$ is the field of contours of f/H and $\hat{z}$ is a unit vertical vector (Rhines 1969*a*, *b*). In the following H is taken to be continuous, although the analysis extends directly to discontinuous profiles provided that the contour field $\boldsymbol{K}$ is uniquely defined everywhere and there are no wall-step junctions present. The difficulties associated with such junctions are discussed in Johnson (1985).

Multiplying (2.1) by Ψ gives the conservation equation

$$E_t + \nabla \cdot \boldsymbol{F} = 0, \tag{2.2}$$

where $E = \frac{1}{2}H^{-1}|\nabla\Psi|^2 = \frac{1}{2}H|\boldsymbol{u}|^2$ is the vertically integrated local kinetic energy density, and the energy flux $\boldsymbol{F}$ can be expressed in various equivalent forms, viz.

$$\boldsymbol{F}_1 = -\frac{\Psi}{H}\{\nabla\Psi_t + f\hat{z} \wedge \nabla\Psi\}, \tag{2.3a}$$

$$\boldsymbol{F}_2 = Hp\boldsymbol{u}, \tag{2.3b}$$

$$\boldsymbol{F}_3 = -\Psi\{H^{-1}\nabla\Psi_t + \tfrac{1}{2}\Psi\boldsymbol{K}\}, \tag{2.3c}$$

where p is the departure of the pressure from equilibrium. These forms differ by quantities whose divergence vanishes. Energy is conserved within regions where the normal component of $\boldsymbol{F}$ vanishes on the boundary. These include regions with solid boundaries or boundaries on which the pressure is constant. Moreover, for time-periodic motions the average value over a period of the net energy flux into any region vanishes.

In a bounded domain $\mathscr{D}$ the inviscid zero-mass-flux condition is

$$\Psi = 0 \quad \text{on } \partial\mathscr{D}, \tag{2.4}$$

where $\partial\mathscr{D}$ denotes the boundary of $\mathscr{D}$. For an unbounded domain such as the semi-infinite channel of Part 2, (2.4) can be replaced by the requirement that disturbances vanish at large distances. The following analysis and results then apply directly to localized modes in such domains. Look for normal mode solutions of (2.1) of the form

$$\Psi = \operatorname{Re}\{e^{-i\sigma t}\psi(x,y)\}. \tag{2.5}$$

For any (sufficiently well behaved) ϕ_0 and ϕ_1 define the inner product

$$\langle\phi_0, \phi_1\rangle = \int_{\mathscr{D}} H^{-1}\nabla\phi_0 \cdot \nabla\phi_1^*, \tag{2.6}$$

where ϕ_1^* is the complex conjugate of ϕ_1. The associated energy functional and norm is

$$E(\phi, \mathscr{D}) = \langle\phi, \phi\rangle. \tag{2.7}$$

Substituting (2.5) in (2.1), multiplying by ψ^* and integrating over $\mathscr{D}$ yields

$$\sigma E(\psi, \mathscr{D}) - \mathrm{i} \int_{\mathscr{D}} \psi^* \boldsymbol{\nabla} \cdot (\psi \boldsymbol{K}) = 0, \tag{2.8}$$

where the divergence theorem and (2.4) have been used to ensure that the first term is positive definite. The equation analogous to (2.8) for σ^* can be written

$$\sigma^* E(\psi, \mathscr{D}) + \mathrm{i} \int_{\mathscr{D}} \psi \boldsymbol{\nabla} \cdot (\psi^* \boldsymbol{K}) = 0. \tag{2.9}$$

Subtracting these equations and using (2.4) gives

$$(\sigma - \sigma^*) E(\psi, \mathscr{D}) = \mathrm{i} \int_{\mathscr{D}} \boldsymbol{\nabla} \cdot (\psi \psi^* \boldsymbol{K}) = \mathrm{i} \int_{\partial \mathscr{D}} \psi \psi^* \boldsymbol{K} \cdot \hat{\boldsymbol{n}} = 0, \tag{2.10}$$

showing the eigenvalues σ to be real (Rhines 1969*a, b*). For a general unbounded domain it is possible that no locally confined modes exist: (2.10) shows that if there are confined modes then their frequencies are real.

Adding (2.8) and (2.9) gives

$$\sigma E(\psi, \mathscr{D}) = F(\psi, \mathscr{D}), \tag{2.11}$$

where for any (sufficiently well behaved) ϕ, the functional F is purely real and takes the forms

$$F(\phi, \mathscr{D}) = \tfrac{1}{2}\mathrm{i} \int_{\mathscr{D}} \{\phi^* \boldsymbol{\nabla} \cdot (\phi \boldsymbol{K}) - \phi \boldsymbol{\nabla} \cdot (\phi^* \boldsymbol{K})\} \tag{2.12a}$$

$$= \mathrm{i} \int_{\mathscr{D}} \phi^* \boldsymbol{\nabla} \cdot (\phi \boldsymbol{K}) - \tfrac{1}{2}\mathrm{i} \int_{\partial \mathscr{D}} \phi \phi^* \boldsymbol{K} \cdot \hat{\boldsymbol{n}}. \tag{2.12b}$$

Using (2.4) allows F to be bounded by

$$|F| = \left| \int_{\mathscr{D}} \frac{f}{H} \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}\phi \wedge \boldsymbol{\nabla}\phi^*) \right| \leqslant f_1 \int_{\mathscr{D}} H^{-1} |\boldsymbol{\nabla}\phi|^2 = f_1 E,$$

where f_1 is the maximum value of $|f|$ in $\mathscr{D}$. In conjunction with (2.11), this shows that $|\sigma| \leqslant f_1$, i.e. normal modes are subinertial.

The largest eigenvalue σ_1 and its associated eigenfunction ψ_1 can be characterized as the solution of the variational problem

$$\sigma = \max_{\phi} F(\phi, \mathscr{D}) / E(\phi, \mathscr{D}) \tag{2.13}$$

subject to the constraint that ϕ vanishes on $\partial\mathscr{D}$. Under this constraint the final term in (2.12*b*) is absent. Moreover, since both E and F are quadratic in ϕ the test functions can be restricted to be of norm unity. Repeating the analysis leading to (2.8), (2.9), (2.10) for eigenfunctions ψ_m and ψ_n with corresponding eigenvalues σ_m and σ_n yields

$$(\sigma_m - \sigma_n) \langle \psi_m, \psi_n \rangle = 0. \tag{2.14}$$

Eigenmodes corresponding to distinct eigenvalues ($\sigma_m \neq \sigma_n$) are orthogonal. The second eigenvalue and its eigenfunction follow from (2.13) by adding the additional constraint that the test function is orthogonal to ψ_1. Higher modes follow similarly. In many applications (e.g. Part 2) the eigenfunctions form a complete set and so a given response g can be expanded as

$$g = \sum_{n=1}^{\infty} a_n \psi_n \quad \text{where} \quad a_n = \langle g, \psi_n \rangle / E(\psi_n, \mathscr{D}). \tag{2.15}$$

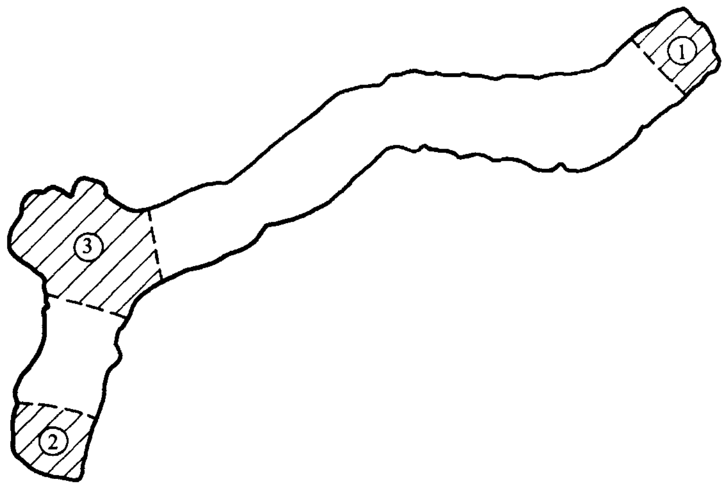

FIGURE 1. An outline of the Swiss Lake of Lugano showing the regions of wave activity for the three gravest topographic modes as found in the finite-element calculations of Trösch (1984).

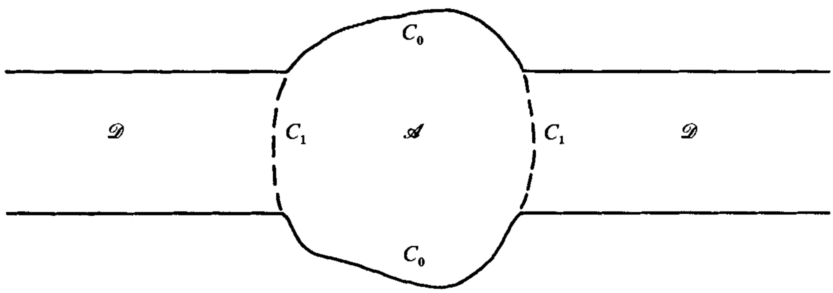

FIGURE 2. The subdomain $\mathscr{A}$ of the full domain $\mathscr{D}$. The curve C_0 consists of those sections of the boundary of $\mathscr{A}$ coincident with the boundary of $\mathscr{D}$ and C_1 consists of the remainder. The text discusses appropriate conditions to be applied at the open boundary C_1.

3. Open boundary conditions

Figure 1 gives an outline of the Lake of Lugano indicating the regions within which strong wave activity is found in the numerical results of Trösch (1984). In general such regions are bounded by a curve that is in part coincident with the lake shore and elsewhere open to the lake. Consider the problem posed in the previous section, let $\mathscr{A}$ be a subdomain of $\mathscr{D}$ and let $\partial\mathscr{A}$ consist of a part C_0 coincident with $\partial\mathscr{D}$ and the remainder C_1 open (figure 2). The totality of eigenfrequencies and eigenfunctions of $\mathscr{D}$ corresponds to those for the subdomains $\mathscr{A}$ and $\mathscr{D}\backslash\mathscr{A}$. In particular if, as in the examples of Part 2, there are no localized modes in $\mathscr{D}\backslash\mathscr{A}$ then there is a one-to-one correspondence between eigenfrequencies of $\mathscr{D}$ and those of $\mathscr{A}$. Alternatively, in the example of a shelf–estuary junction in Stocker & Johnson (1989*b*) where localized modes are possible in the domain $\mathscr{D}\backslash\mathscr{A}$ also, both sets of modes are required to give the totality of modes in $\mathscr{D}$. The correspondence is between the merged series in descending order of the frequencies of the subdomains and the frequencies in descending order of $\mathscr{D}$. Further discussion of this point is given in Stocker & Johnson (1989*b*). To simplify the numbering and in line with Part 2 the results below are for the case where $\mathscr{D}\backslash\mathscr{A}$ does not support localized modes. Then a lower bound for σ_1 is given by

$$\sigma_1^{\ell} = \max_{\phi} F(\phi, \mathscr{A})/E(\phi, \mathscr{A}), \tag{3.1}$$

subject to the constraint that ϕ vanishes on $\partial\mathscr{A}$, i.e. the zero-mass-flux conditions are applied over both C_0 and C_1. Let ψ_1^{ℓ} be the corresponding eigenfunction. That σ_1^{ℓ} is a lower bound for σ_1 follows by noting that the function equal to ψ_1^{ℓ} in $\mathscr{A}$ and vanishing in $\mathscr{D}\backslash\mathscr{A}$ is an admissible function for (2.13) with the same values of F and E as on the restricted domain.

Let $\sigma_1^{\mathrm{u}}, \psi_1^{\mathrm{u}}$ be the solution of the variational problem (3.1) where the trial functions ϕ are subject solely to the constraint that they vanish on C_0. Now ψ_1^{u} satisfies (2.1) in $\mathscr{A}$ with eigenvalue σ_1, and ψ_1^{u} restricted to $\mathscr{A}$ is an admissible function for the partially unconstrained problem; thus $\sigma_1 < \sigma_1^{\mathrm{u}}$. On the open boundary sections comprising C_1, ψ_1^{u} satisfies the natural boundary condition of the variational problem. This follows from (3.1) and the definitions of E and F as

$$(2\mathrm{i}\sigma H^{-1}\nabla\psi - \psi\boldsymbol{K})\cdot\hat{\boldsymbol{n}} = 0, \tag{3.2}$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to C_1. Comparison with (2.3*c*) shows that (3.2) requires the energy flux $\boldsymbol{F}_3\cdot\hat{\boldsymbol{n}}$ to vanish at each point on C_1, a stronger requirement than that for periodic motion (i.e. that $\int_{C_1}\boldsymbol{F}\cdot\hat{\boldsymbol{n}}$, the total flux across C_1, vanishes). In terms of normal (∂n) and tangential (∂s) derivatives (3.2) can be written

$$\frac{\partial^2\Psi}{\partial n\,\partial t} - \tfrac{1}{2}H\frac{\partial}{\partial s}\left(\frac{f}{H}\right)\Psi = 0. \tag{3.3}$$

If the boundary C_1 coincides with a contour of f/H then (3.3) reduces to the requirement that the normal derivative vanishes, i.e. that the tangential velocity is zero on the open boundary.

Since the eigenmodes of the three variational problems are not mutually orthogonal, the demonstration that the formulations leading to σ_1^{ℓ} and σ_1^{u} also provide bounds for higher eigenvalues is not immediate. Such a demonstration follows however from the maximum–minimum description for higher eigenvalues (Courant & Hilbert 1953) and adapting arguments for subdomains of vibrational problems. Thus if σ_n^{ℓ} are the natural frequencies obtained by solving (2.1) on the restricted domain $\mathscr{A}$ subject to the boundary condition that ψ vanishes on $\partial\mathscr{A}$, and σ_n^{u} are the frequencies obtained by solving (2.1) subject to ψ vanishing on C_0 and (3.2) holding on C_1, then

$$\sigma_n^{\ell} < \sigma_n < \sigma_n^{\mathrm{u}}. \tag{3.4}$$

The reduced problems are closely related to the two boundary conditions commonly applied when modelling rectilinear shelves abutting open ocean regions. The usual boundary condition applied at the outer edge of the shelf is the vanishing of either the mass flux, leading to a lower bound on the frequency, or the tangential velocity, leading to an upper bound.

An estimate of the frequency of the mode on the whole domain follows by solving directly the two problems on the reduced domain and forming the average $\tfrac{1}{2}(\sigma_n^{\ell}+\sigma_n^{\mathrm{u}})$. The maximum theoretical error in the estimate is $\tfrac{1}{2}(\sigma_n^{\mathrm{u}}-\sigma_n^{\ell})$ although in the examples discussed below it is far smaller. The estimates give close agreement with calculated results for the full problem in the example of a semi-infinite channel with a bay discussed in Part 2. For the problem of an elliptical basin discussed in Mysak

	$m = 1$				$m = 2$			
	σ^{ℓ}_{1n}	σ_{1n}	σ^{e}_{1n}	σ^{u}_{1n}	σ^{ℓ}_{2n}	σ_{2n}	σ^{e}_{2n}	α^{u}_{2n}
$n = 1$	0.1587	0.1728	0.1714	0.1841	0.2440	0.2491	0.2491	0.2541
$n = 2$	0.03614	0.03725	0.03706	0.03798	0.06696	0.06765	0.06763	0.06829
$n = 3$	0.01451	0.01470	0.01466	0.01482	0.02808	0.02822	0.02822	0.02835

TABLE 1. The upper and lower bounds σ^{u}_{mn} and σ^{ℓ}_{mn}, estimates $\sigma^{e}_{mn} = \frac{1}{2}(\sigma^{\ell}_{mn} + \sigma^{u}_{mn})$ and exact values σ_{mn} of the eigenfrequencies of modes round an axisymmetric island with a skirt. The island has radius unity, the skirt is contained in a disk of radius $a = 2$ and the outer boundary of the reduced domain is at $R = 2a$.

(1985), Mysak *et al.* (1985) and Johnson (1987), bounds for the full solution are given by solving the reduced problem with zero mass flux over the centreline (i.e. σ^{ℓ}_{n}) and with the tangential velocity zero there (i.e. σ^{u}_{n}), the solution towards which the unconstrained variational principle of Mysak (1985) converges.

As a further simple example consider the modes of an f-plane (taking $f \equiv 1$) for an axisymmetric island with a skirt:

$$H = \begin{cases} H_0 r^{2b} & (1 \leqslant r \leqslant a) \\ H_0 a^{2b} & (r \geqslant a), \end{cases} \tag{3.5}$$

for constants a, b, H_0 and radius r. Let full domain $\mathscr{D}$ be the region $r \geqslant 1$ and the reduced domain $\mathscr{A}$ be the annulus $1 \leqslant r \leqslant R$ for some $R \geqslant a$. Then the eigenfrequencies for each of the three problems can be written

$$\sigma = -2mb/(\lambda^2 + m^2 + b^2), \tag{3.6}$$

where m is the (integral) azimuthal wavenumber and the radial wavenumber λ satisfies

$$\frac{b + \lambda \cot(\lambda \log a)}{m} = \begin{cases} -\left[1 + \left(\frac{a}{R}\right)^{2m}\right] \Big/ \left[1 - \left(\frac{a}{R}\right)^{2m}\right] & \text{for } \sigma^{\ell}_{mn} \\ -1 & \text{for } \sigma_{mn} \\ -\left[1 - \left(\frac{a}{R}\right)^{2m}\right] \Big/ \left[1 + \left(\frac{a}{R}\right)^{2m}\right] & \text{for } \sigma^{u}_{mn}. \end{cases} \tag{3.7}$$

Table 1 gives values of the frequencies for $a = 2$, $b = 1$, $R = 2a$. The accuracy of the estimates is worst for the lowest modes but increases rapidly with azimuthal wavenumber. Even for the fundamental radial mode at $m = 2$ the estimate gives four significant figures. The increase in accuracy with increasing n is less dramatic. Figure 3 shows the variation in the bounds and estimate for σ_{11} with increasing R/a. The estimate is graphically indistinguishable from the exact value before R exceeds $3a$.

4. Discussion

It has been shown that upper and lower bounds on the frequencies of normal modes on large domains can be obtained by considering two distinct conditions at the

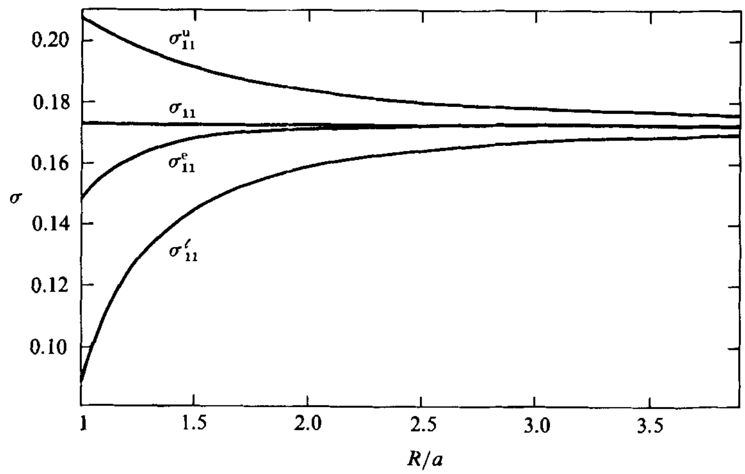

FIGURE 3. The bounds σ^ℓ_{11}, σ^u_{11}, estimate σ^e_{11} and exact value σ_{11} for the frequency of the fundamental mode about an island with a skirt plotted as a function of the radius R of the circle forming the open boundary.

open boundaries of suitably chosen subdomains. A lower bound is obtained by requiring the mass flux across the boundary to vanish and an upper bound by requiring that a particular form of the energy flux should vanish at each point on the open boundary. If f/H is constant along the boundary this latter condition reduces to requiring the tangential velocity to vanish.

In numerical computations such as those of Trösch (1984), restricting the domains to the various bays would allow the details of local topography to be more closely modelled. The separation of the bounds then gives a measure of the error introduced by ignoring the remainder of the domain. From the examples presented here and those in Part 2 it is evident that the bounds become increasingly accurate for higher modes, most rapidly for those with more structure along the open boundary, and are at their worst for the fundamental. By taking the open boundary far enough from the localized motion, the boundary conditions are almost satisfied, so the upper and lower bounds converge to the true value. Higher modes tend to be better confined and hence better estimated.

Equation (2.1) applies exactly to wave motion in homogeneous shallow flow ($H \ll L$, for horizontal scale L) if the surface is rigid. No restriction is placed on the size of depth changes and so (2.1) finds much use in modelling the dynamics on shelves and in lakes, shallow seas, bays and estuaries. In these applications the largest deviations from the model requirements are due to free-surface deformation and stratification.

The importance of the free surface in subinertial modes is measured by the non-dimensional Rossby radius $d = (gH)^{\frac{1}{2}}/fL$ (for gravitational acceleration g). If $d \gg 1$ the surface is effectively rigid and (2.1) is accurate. If d is of order unity or less then (2.1) provides at best a qualitative description of the flow. Typical values of the Rossby radius vary from 200 km for mid-latitude shallow seas or lakes with depths of order 40 m, to 2000 km for oceans with depths of order 4 km, the scales over which (2.1) is a reasonable model of low-frequency behaviour. When the surface is free additional sets of normal modes with superinertial frequencies occur corresponding to Kelvin and Poincaré waves. The full equations can be analysed by the methods used here but the results are less straightforward except in two low-frequency limits.

First, if attention is confined to small fractional depth changes then at low frequencies only free-surface modified topographic waves are present, an additional potential energy term appears in E but the zero-mass-flux, natural boundary conditions and frequency bounds are unaltered. Secondly, if depth changes are slowly varying but not necessarily small then at low frequencies Kelvin waves are present but not Poincaré waves. Again it appears that simple useful open boundary conditions and frequency bounds can be obtained.

Stratification alters the behaviour of the flow markedly, allowing extra, internal waves. Extension of the present results to domains where stratification is important is not straightforward. However, for many applications, as in the two-layer model of the Lake of Lugano in Mysak *et al.* (1985), the dominant normal mode frequencies are determined by the barotropic dynamics.

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